Optimization of Redundancy Selection in the Finite-Element Force Method

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Techniques for the automatic selection of redundancies in the matrix force method are well established. However, optimization of the selection is an area which still requires considerable attention. The criterion for optimum selection requires the incorporation of the relative element stiffnesses by a transformation of the variables and a suitable choice of pivots in the reduction of the transformed equilibrium equations. Two forms of weighting and a refined method of pivot selection have been developed in this paper. The incorporation of rigid elements is briefly discussed. The results of a numerical comparison between the various forms of weighting and pivot selection, including previously published methods, are presented for a series of frame and beam problems.

Introduction

THE finite-element force method of redundant analysis became fully automated with the introduction of techniques for the automatic selection of redundancies. These procedures, namely the Rank Technique and the Structure Cutter, are presented in detail in Refs. 1–3.

Optimum selection of redundancies[4] depends on the relative stiffnesses of the individual structural elements. Methods of accounting for the effect of relative stiffnesses are given by Denke[3] and Contini and Haggenmacher.[5] This paper presents the method of Ref. 5 as well as a new method based on sounder theoretical principles.

All approaches use some method of transforming the independent generalized force variables and, therefore, the coefficient matrix in the equilibrium equations. Denke used the diagonal coefficients of the assembled element stiffness matrix and Contini and Haggenmacher the square roots of those coefficients as scalar factors applied to the independent generalized force variables. The use of the square roots was prompted by considerations of dimensional consistency. To remove the approximation involved in neglecting the off-diagonal terms of the stiffness matrix, Robinson expressed the matrix as the product of a lower triangular matrix, obtained by means of a Cholesky decomposition, and its transpose. This technique retains the dimensional consistency of the square root approach and results in transformed variables which are linear combinations of the original variables within each structural element.

All these methods break down if some of the structural elements are rigid, because the assembled element flexibility matrix is singular. It is necessary, therefore, to assign suitable finite stiffnesses to the rigid elements in a rational manner.

Having transformed the equilibrium equations, a reduction is made using an elimination procedure such as the Jordanian. In such a procedure, the selection of the pivot, from the current coefficient matrix, critically determines the statically determinate substructure and, consequently, the redundancies. It is essential to a meaningful pivot choice that all coefficients in each row of the coefficient matrix be dimensionally identical. In the past, the pivot has been chosen as the largest absolute value in either each consecutive row or the whole remaining matrix. It was suggested by Contini that the choice of the pivot should be based on the relative dominance of the largest absolute value in each row. This method is referred to as the "distribution technique" and is discussed in the text.

Equilibrium Equations

In the redundant force method of structural analysis a system of joint equilibrium equations is generated which relates the independent generalized force variables $\{F\}$ and the discrete generalized applied nodal loads $\{L\}$. This system of equations can be written in matrix form as

$$[E]\{F\} + \{L\} = \{O\} \tag{1}$$

or

$$\{L\} = -[E]\{F\} \tag{2}$$

Transformation of Variables

When a structure is redundant, the forces in the structural elements cannot be found by equilibrium considerations only. The matrix $[E]$ then consists of a square nonsingular submatrix, the columns of which pertain to element forces which can be statically determined, and a remaining submatrix, usually rectangular, whose columns correspond to those element forces that are selected as redundants. The redundancies should be chosen in such a way that the difference between the determinate solution and the indeterminate solution is a minimum. This criterion states in effect that the determinate structure should provide the stiffest possible loadpath. Therefore, the relative flexibilities or stiffnesses of the individual structural elements are an essential ingredient in any procedure for the automatic selection of redundancies.

Denke[3] uses the direct element stiffness factors $[k_{ii}]$ for weighting of the coefficient matrix $[E]$. However, only if the independent generalized force variables are of the same dimensions does this type of weighting lead to a dimensionally consistent weighted $[E]$ matrix, and hence a meaningful pivot selection. This will be seen from the following developments which lead to a dimensionally correct weighting under all circumstances.

The generalized element force variables and their corresponding generalized element deformations $\{d\}$ can be re-

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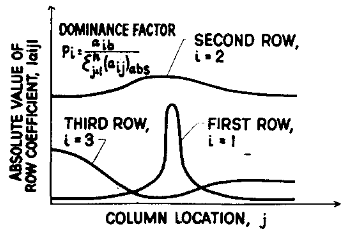

Fig. 1 Distribution technique, row distributions.

lated using the assembled element stiffness matrix $[k]$, that is,

$$\{F\} = [k]\{d\} \tag{3}$$

The internal energy in the structure is given by

$$U = \tfrac{1}{2}\{F\}^T\{d\} \tag{4}$$

From nodal compatibility considerations the element deformations and the structural displacements $\{r\}$ can be related by the simple transformation,

$$\{d\} = [a]\{r\} \tag{5}$$

The work done by the external loads is given by

$$W = \tfrac{1}{2}\{L\}^T\{r\} \tag{6}$$

Therefore, equating the internal and external energies,

$$\{L\}^T\{r\} = \{F\}^T\{d\} \tag{7}$$

Substituting from Eqs. (2) and (5) into Eq. (7) yields

$$-\{F\}^T[E]^T\{r\} = \{F\}^T[a]\{r\}$$

and, hence,

$$[a] = -[E]^T \tag{8}$$

Therefore, using Eq. (2) and substituting from Eqs. (3, 5, and 8),

$$\{L\} = [E][k][E]^T\{r\} \tag{9}$$

or

$$\{L\} = [K]\{r\}$$

where the constrained structural stiffness matrix is given by

$$[K] = [a]^T[k][a] = [E][k][E]^T \tag{10}$$

Equations (9) and (10) give a formulation of the well-known displacement method.

The assembled element stiffness matrix is symmetrical, positive definite, and of narrow band width. For the very simple structural elements, the stiffness matrices are scalars, and the $[k]$ matrix can be written as

$$[k] = [(k_{ii})^{1/2}][(k_{ii})^{1/2}] \tag{11}$$

When an element stiffness matrix is nonscalar, an approximation can be made using only the diagonal terms of $[k]$. Equation (10) then becomes

$$[K] = [E][(k_{ii})^{1/2}][(k_{ii})^{1/2}][E]^T \tag{12}$$

This indicates that, in order to account for the relative-element flexibilities or stiffnesses, the equilibrium equations should be weighted using the square root of the diagonal terms of the assembled element stiffness matrix.

Defining a weighting matrix $[W_1] = [(k_{ii})^{1/2}]$, Eq. (1) can now be written as

$$[E][W_1][W_1]^{-1}\{F\} + \{L\} = \{O\}$$

or

$$[A]\{P\} + \{L\} = \{O\} \tag{13}$$

where

$$\{P\} = [W_1]^{-1}\{F\} \tag{14}$$

and

$$[A] = [E][W_1] \tag{15}$$

The transformations expressed by Eqs. (14) and (15) not only perform the required weighting but also guarantee equal dimensions in each row of $[A]$ and all rows of $\{P\}$, regardless of the choice of generalized force variables. The original variables are then calculated by solving Eq. (14), that is,

$$\{F\} = [W_1]\{P\} \tag{16}$$

It should be realized that the element stiffness matrices used in this weighting procedure are obtained by inverting the element-flexibility matrices. If the flexibility matrix for structural element i is denoted by $[D_i]$, then,

$$[k_i] = [D_i]^{-1} \tag{17}$$

The assembled element flexibility matrix is denoted by $[D]$.

An approach will now be presented which removes the need for the approximation implied by Eq. (11). This will establish a formal method of incorporating the relative elastic properties of the structural elements for the process of automatically selecting redundancies. Since the assembled element stiffness matrix is symmetrical and positive definite, it can be written as

$$[k] = [k_L][k_L]^T \tag{18}$$

where $[k_L]$ is a lower-triangular matrix (obtained using Cholesky's decomposition procedure).

Usually, the individual element stiffness matrices in the assembled element stiffness matrix are uncoupled. In this case, the lower triangular matrix can be evaluated by simply applying the Cholesky decomposition to each individual-element stiffness matrix separately. Equation (10) can now be written as

$$[K] = [E][k_L][k_L]^T[E]^T \tag{19}$$

Therefore, the equilibrium equations should be weighted using a lower triangular matrix derived from the assembled-element stiffness matrix. This introduces not only the influence of the diagonal stiffness coefficients but also the influence of the cross-coupling coefficients which occur among certain independent generalized force variables. Similar expressions to Eqs. (14–16) can be obtained by substituting $[W_2] = [k_L]$ for $[W_1]$.

From Eq. (14) it can be seen that the independent generalized force variables have been transformed into a new set of variables $\{P\}$. When using $[W_1]$, the original variables are directly factored, using the corresponding $(k_{ii})^{1/2}$ value, to

Table 1 Circular frame problems

Problem	Properties
1	Constant $A = 1.0$ in.^{2}, constant $I = 5.0$ in.^{4}
2	Constant $A = 1.0$ in.^{2}, constant $I = 20.0$ in.^{4}
3	Constant $A = 1.0$ in.^{2}, constant $I = 80.0$ in.^{4}
4 & 5	Nodes 1 and 19, $A = 0.2$ in.^{2}, $I = 1.0$ in.^{4} Nodes 10 and 28, $A = 1.0$ in.^{2}, $I = 5.0$ in.^{4} Linear variation of A and I
6	Varying A, as problem 4 constant $I = 5.0$ in.^{4}
7	Varying A, as problem 4 constant $I = 20.0$ in.^{4}
8	Varying A, as problem 4 constant $I = 80.0$ in.^{4}
9	Constant $A = 1.0$ in.^{2} Elements 3–4, 5–6, 15–16, 17–18, 27–28, 29–30, $I = 2.0$ in.^{4} Elements 4–5, 16–17, 28–29, $I = 1.0$ in.^{4} All other elements, $I = 5.0$ in.^{4}

give the transformed variables. However, when using $[W_2]$, the transformed variables are actually combinations of the original variables within an element.

In the weighting approaches described, it has been assumed that all the structural elements are elastic. To account for rigid elements, the aim is to make sure that they have a greater weighting effect than the elastic elements. To meet this requirement, rigid elements are then assigned values considerably greater than the largest elastic weighting factor.

Investigation of Transformed Equilibrium Equations

The transformed system of equilibrium equations is given by

$$[A]\{P\} + \{L\} = \{O\} \tag{20}$$

This system of equations is now investigated using the rank technique which adopts the Jordan elimination procedure. If matrix $[A]$ is of order $(m \times n)$, then the final equivalent matrix will contain m columns which contain one nonzero coefficient (equal to unity). These columns correspond to the locations of the pivot elements used in the elimination procedure. The transformed variables corresponding to the remaining columns constitute the automatically selected redundancies. The investigated system of equations can now be expanded[6] to give

$$\{P\} = [B_0]\{L\} + [B_1]\{X\} \tag{21}$$

In this equation, the transformed variables are expressed explicitly in terms of the redundancies $\{X\}$ and the applied loads. These redundancies are referred to as "transformed redundancies." Matrices $[B_0]$ and $[B_1]$ are usually referred to as the basic and redundant load transformation matrices, respectively. It should be emphasized that these matrices are automatically generated in the process of automatically selecting the redundancies.

Premultiplying both sides of Eq. (21) by the weighting matrix $[W]\{(k_{ii})^{1/2}$ or $[k_L]\}$ gives

$$\{F\} = [W]\{P\} = \{F_0\} + [f]\{X\} \tag{22}$$

where

$$\{F_0\} = [W][B_0]\{L\} \tag{23}$$

and

$$[f] = [W][B_1] \tag{24}$$

Therefore, the redundant system of independent generalized force variables corresponding to a unit transformed redundancy is given by the respective column in matrix $[f]$.

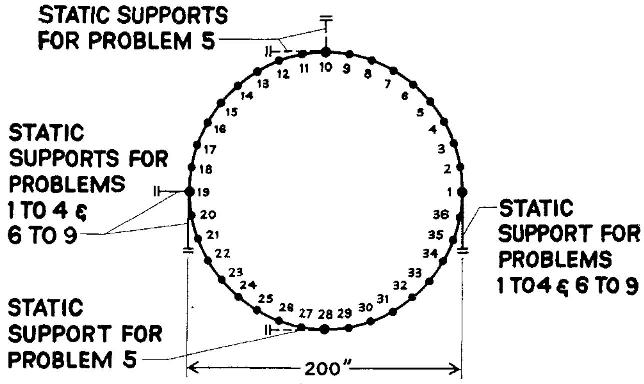

Fig. 2 Circular frame.

Table 2 Circular frame problems

Problem	Investigation procedure		Conditioning parameter ϵ^* for various weightings: No weighting	$[(k_{ii})^{1/2}]$	$[k_L]$	$[k_{ii}]$
1	RANTEC	1	0.0199	0.0785	0.1537	0.7405
	RANTEC	2	0.5000	0.5000	0.7785	0.9400
	RANTEC	3	0.4886	0.9800	1.000	0.7790
2		1	0.0199	0.1675	0.3780	0.7405
		2	0.5000	0.9198	0.7570	0.8560
		3	0.4886	0.9416	0.6218	0.7789
3		1	0.0199	0.1671	0.6013	0.2583
		2	0.5000	0.7554	0.6609	0.3297
		3	0.4886	0.7785	0.8658	0.2615
4		1	0.0339	0.5791	0.4452	0.1422
		2	0.6770	0.9207	0.7907	0.1422
		3	0.3228	0.6837	0.6354	0.1422
5		1	0.4213	0.4440	0.2341	0.1320
		2	0.3230	0.6376	0.6748	0.1320
		3	0.6711	0.9168	0.6748	0.1320
6		1	0.0199	0.1675	0.3273	0.7407
		2	0.5000	0.8231	0.9796	0.7553
		3	0.4884	0.9200	0.7786	0.7792
7	RANTEC	1	0.0199	0.7413	0.6023	0.2458
	RANTEC	2	0.5000	0.9421	0.8667	0.0431
	RANTEC	3	0.4879	0.7797	0.6023	0.6573
8		1	0.0200	0.7434	0.6041	0.0884
		2	0.5000	0.9205	0.6639	0.0438
		3	0.4858	0.7816	0.7811	0.2917
9		1	0.0199	0.9798	0.9797	0.9999
		2	0.5000	0.9798	0.9999	0.9798
		3	0.4886	0.9999	0.9999	0.9798

Jordan Elimination Procedure

The transformed system of equilibrium equations will be rewritten in a slightly modified form,

$$[A]\{P\} + [I]\{L\} = \{O\} \tag{25}$$

This system of equations can now be investigated by applying the Jordan elimination procedure to the augmented matrix $[A:I]$. Before commencing the investigation it is desirable that the coefficients in each row of matrix $[A]$ be dimensionally identical. Weighting of the original system of equilibrium equations using either of the matrices $[W_1]$ and $[W_2]$ achieves this purpose. In the Jordan elimination procedure, the general step is to locate a suitable pivot in the current equivalent coefficient matrix $[A]$ from a row and column not previously used for pivoting. Denote a particular pivot by a_{pq} and normalize the pth row with respect to it. The qth coefficient in each of the remaining rows is now reduced to zero. This is achieved by subtracting from each remaining row i, the normalized row p, multiplied by a_{iq}. The qth column will then contain a unit value in the pth row and zero in all other rows. The whole process is continued until all rows have been exhausted.

Choice of Pivot

There are several alternative ways of choosing pivots. A pivot could be chosen as the largest absolute value in a row, taking the rows consecutively, or the largest absolute value in the current equivalent coefficient matrix. However, a more refined pivot selection routine will now be described. This will be referred to as the "distribution technique." When the coefficients in each row of matrix $[A]$ are dimensionally identical, a distribution plot could be made which shows the pivot dominance in a particular row. Typical row distributions are shown in Fig. 1. The largest absolute

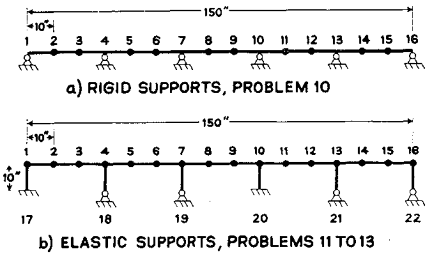

Fig. 3 Continuous beams.

value occurs in row 2, however its dominance compared with the other coefficients in the row is not as significant as in row 1. A dominance factor, therefore, will be established; this is given by

$$p_i = \left(a_{ib} \Big/ \sum_{j=1}^{n} (a_{ij})_{abs} \right) \tag{26}$$

where a_{ib} is the largest absolute value in row i, and n is the number of columns in the coefficient matrix.

The first step in the distribution technique is to evaluate the dominance factor for each row in matrix $[A]$. The pivot chosen for the elimination is the one in the row with the largest dominance factor. The whole distribution technique is applied to each current equivalent coefficient matrix.

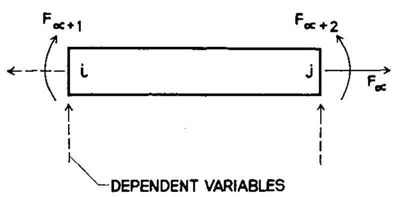

Fig. 4 Independent generalized force variables for beam element.

Numerical Comparison of Methods

To assess the methods of weighting and pivot selection, a series of problems was studied. The details of these are shown in Figs. 2–4 and Tables 1 and 3. For this purpose, three procedures for the reduction of Eq. (25) were written as computer subroutines using different methods of pivot selection. These were designated as follows: 1) RANTEC 1, pivot selection by scanning the next row only; 2) RANTEC 2, pivot selection by scanning the complete remaining matrix; 3) RANTEC 3, pivot selection using the distribution technique.

With each of the pivot selection routines, the following weightings were carried out: 1) no weighting, 2) $[(k_{ii})^{1/2}]$, 3) $[k_L]$, 4) $[k_{ii}]$.

To compare the various combinations of pivot selection and weighting, the following criterion was adopted. The ultimate test of the selected redundancies is the conditioning of the coefficient matrix in the continuity equations,

Table 3 Continuous beam problems

Problem	Properties
10	Constant $A = 1.0$ in.², constant $I = 10.0$ in.⁴ Element 1–17, $A = 10.0$ in.², $I = 1.0$ in.⁴ Elements 4–18, 13–21, $A = 10.0$ in.², $I = 0.25$ in.⁴
11	Elements 7–19, 16–22, $A = 10.0$ in.², $I = 0.5$ in.⁴ Element 10–20, $A = 10.0$ in.², $I = 5.0$ in.⁴ All other elements, $A = 1.0$ in.², $I = 5.0$ in.⁴
12	As problem 11, except for elements 1–17, 10–20, $I = 0.1$ in.⁴
13	As problem 11, except for element 1–17, $I = 50.0$ in.⁴

$$[V]\{X\} + \{V_0\} = \{O\} \tag{27}$$

where

$$[V] = [f]^T[D][f] \tag{28}$$

and

$$\{V_0\} = [f]^T[D]\{F_0\} \tag{29}$$

The redundancies are given by

$$\{X\} = -[V]^{-1}\{V_0\} \tag{30}$$

Since the best conditioned matrix for inversion is a diagonal one, the following parameter is adopted as a practical means to measure the "conditioning" of matrix $[V]$. Define

$$\det[V] - \det[V_{ii}] = \epsilon \tag{31}$$

where $[V_{ii}]$ is a diagonal matrix consisting of the diagonal terms of matrix $[V]$.

The value ϵ should approach zero for an ideally conditioned matrix. Therefore, the following conditioning parameter, ϵ^*, will be adopted:

$$\epsilon^* = \det[V]/\det[V_{ii}] \tag{32}$$

For an ideally conditioned matrix ϵ^* should approach unity.

This criterion was chosen (many others exist) because it is simple, a single number, and has some practical significance. The aim of the paper is not to compare different criteria but to show the improvement in results relative to various weighting procedures. The area of matrix conditioning is a very delicate subject and would require an independent study to do the subject justice.

The results for nine circular frame problems are shown in Table 2, and Figs. 5 and 7, and for four continuous beam problems in Table 4 and Fig. 6.

Conclusions

From these efforts to obtain the optimum redundancy selection, the following main conclusions were drawn.

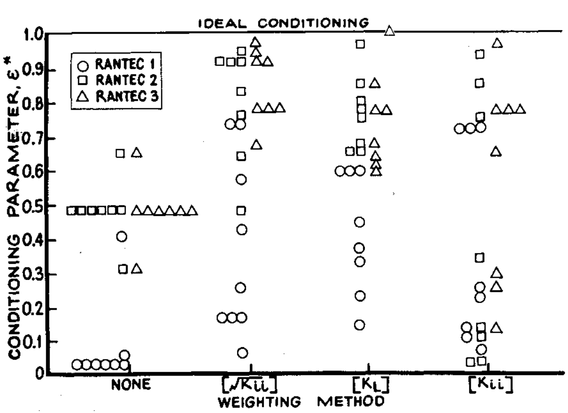

Fig. 5 Circular frames, problems 1–8.

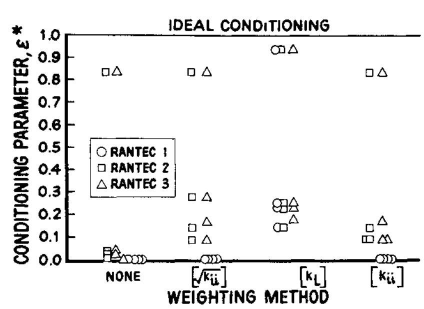

Fig. 6 Continuous beams, problems 10–13.

Table 4 Continuous beam problems

Problem	Investigation procedure	Conditioning parameter ϵ^* for various weightings			
		No weighting	$[(k_{ii})^{1/2}]$	$[k_L]$	$[k_{ii}]$
10	RANTEC 1	0.199×10^{-3}	0.199×10^{-3}	0.9443	0.199×10^{-3}
	RANTEC 2	0.8164	0.8164	0.9443	0.8164
	RANTEC 3	0.8164	0.8164	0.9443	0.8164
11	1	0.2915×10^{-9}	0.2647×10^{-12}	0.2523	0.2646×10^{-12}
	2	0.0132	0.0951	0.2523	0.0951
	3	0.0132	0.0951	0.2523	0.0951
12	1	0.1707×10^{-8}	0.6496×10^{-15}	0.1381	0.6496×10^{-15}
	2	0.2186×10^{-2}	0.1473	0.1381	0.1473
	3	0.2186×10^{-2}	0.1594	0.1674	0.1594
13	1	0.3575×10^{-9}	0.1673×10^{-11}	0.2357	0.1534×10^{-13}
	2	0.0396	0.2788	0.2357	0.0885
	3	0.0396	0.2788	0.2357	0.0885

1) The columns of the coefficient matrix in the equilibrium equations should be weighted to take account of the relative stiffnesses of the structural elements.

2) All the coefficients of each row of the resulting transformed matrix must be dimensionally identical.

3) The $[(k_{ii})^{1/2}]$ and $[k_L]$ type weightings and the pivot selections of RANTEC 2 and 3 were shown to be the most reliable. These selection procedures are unaffected by the arrangement of equations and variables. No significant differences between these weighting and pivot selection methods are shown by the present set of problems. In the continuous beam problems, the pivot selection of RANTEC 1 resulted in a reasonable selection only in conjunction with the $[k_L]$ weighting. This indicates that this type of weighting is the more reliable, as would be expected because of its sound theoretical background. The relative attributes of these methods should be more pronounced when using complex structural elements.

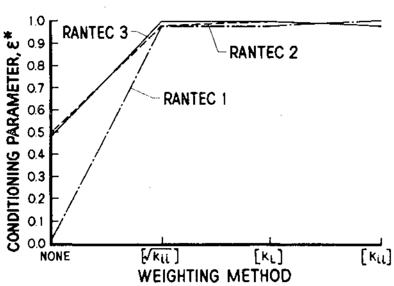

Fig. 7 Circular frame, problem 9.

4) The selection of pivots from consecutive rows is unreliable. This is unfortunate since it would result in the more efficient programing procedure for the analysis of large practical structures.

5) In the $[k_L]$ type weighting the transformed variables are combinations of the force variables within each element. This approach leads to a better redundancy selection than previously, but will not yet reach the optimum of orthogonal systems. The next step appears to be to devise a systematic procedure of variable transformation which combines a larger series of force variables.

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